CHAPTER 7

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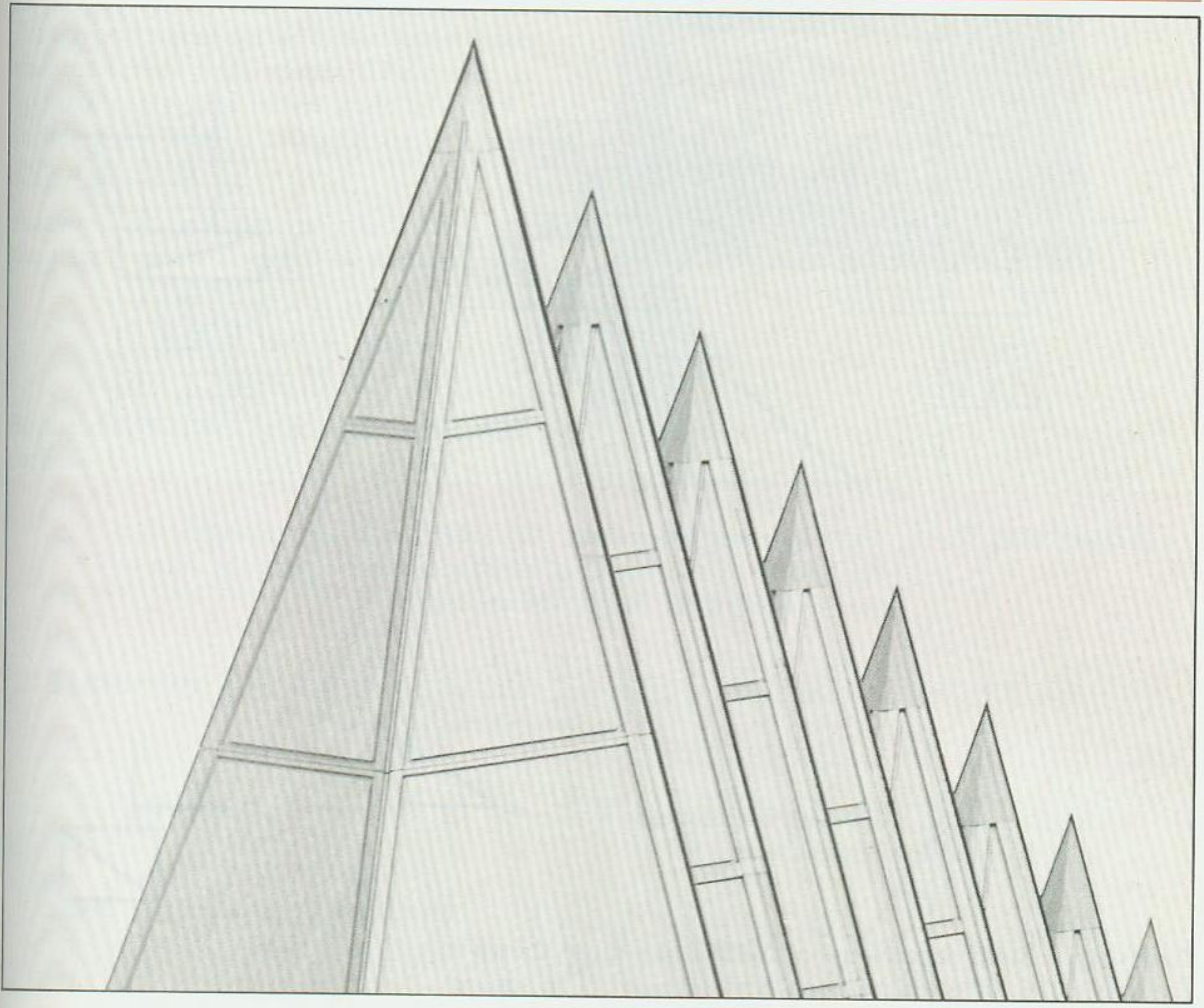
More on Triangles



7–1 The Pythagorean Theorem

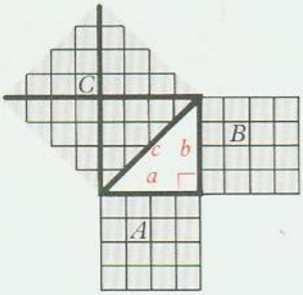
One of the best known and most useful theorems in plane geometry is the Pythagorean Theorem, named after the Greek mathematician Pythagoras.

The theorem says that the area of the square built upon the hypotenuse of a right triangle is equal to the sum of the areas of the squares built upon the legs of the triangle.

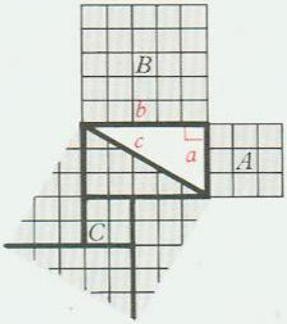
REVIEW: The definitions of right triangle, hypotenuse, and legs of a right triangle, page 199.

In Examples 1–3, find a way of counting the tiny unit squares to show that the area of squares A and B equals the area of the square C on the hypotenuse.

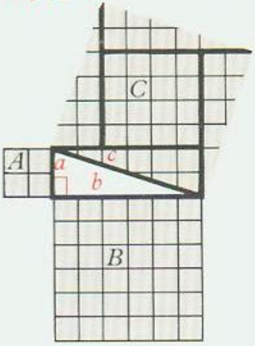
Example 1



Example 2

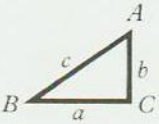


Example 3



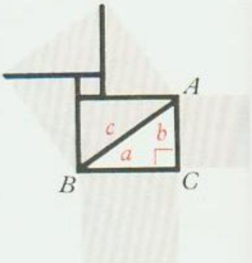
Theorem 7–1 **Pythagorean Theorem.** If $\triangle ABC$ is a right triangle, then the square of the length of the hypotenuse equals the sum of the squares of the lengths of the legs.

PROOF

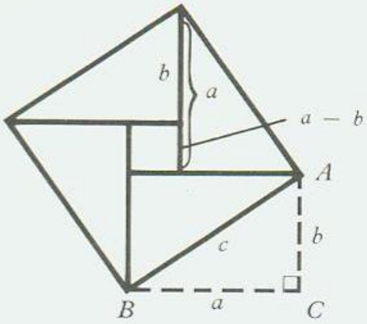


Given: Right triangle ACB with hypotenuse length c and leg lengths a and b

Prove: $c^2 = a^2 + b^2$



Analysis: Build squares upon $\triangle ABC$ like those shown in Examples 1–3. The square upon a has area a^2. The square upon side b has area b^2. The square upon c has area c^2.



The square upon side c consists of four triangles congruent to $\triangle ABC$ and a square. The figure shows that the length of the side of the small square is $a - b$. We can find the area of the large square by adding the areas of the four triangles to the area of the small square. The area of the triangles is $\frac{1}{2}ab$. The area of the square is $(a - b)^2$. So

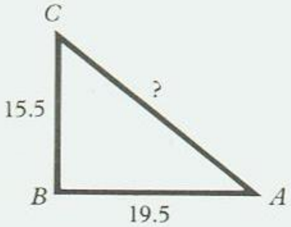
$$\begin{aligned} c^2 &= 4(\tfrac{1}{2}ab) + (a - b)^2 \\ &= 2ab + (a^2 - 2ab + b^2) = a^2 + b^2. \end{aligned}$$

APPLICATION

An advertisement for a television set states that the screen is 25 inches. The screen is approximately 19.5 inches wide and 15.5 inches high. Why is it legitimate to advertise a 25-inch screen?

Answer. The manufacturer is actually stating the length of the diagonal of the screen.

$$\begin{aligned} AB^2 + BC^2 &= AC^2 \\ 19.5^2 + 15.5^2 &= 620.5 \\ AC &\doteq 24.9 \end{aligned}$$



So the diagonal of the screen is almost 25 inches.

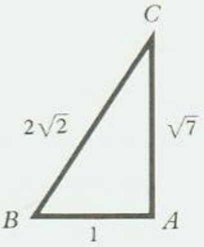
The next theorem is the converse of the Pythagorean Theorem.

Theorem 7–2 If $\triangle ABC$ has side lengths a, b, and c and $c^2 = a^2 + b^2$, then $\triangle ABC$ is a right triangle.

Example

$\triangle ABC$ is a right triangle because

$$(\sqrt{7})^2 + 1^2 = (2\sqrt{2})^2. \qquad (7 + 1 = 8)$$

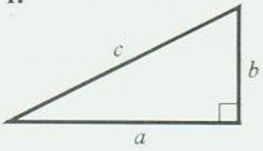


EXERCISES

A.

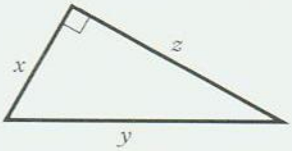
For exercises 1–6 state whether or not the given equation is correct.

1.



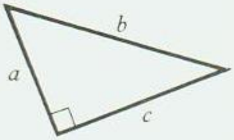
$c^2 = a^2 + b^2$

2.



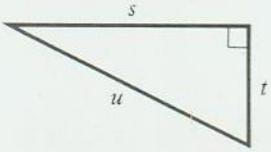
$x^2 + y^2 = z^2$

3.



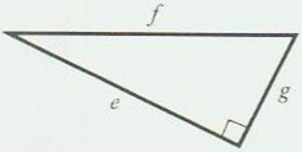
$c^2 = a^2 + b^2$

4.



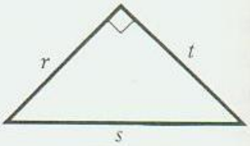
$s^2 = u^2 - t^2$

5.



$f = \sqrt{e^2 + g^2}$

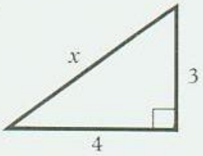
6.



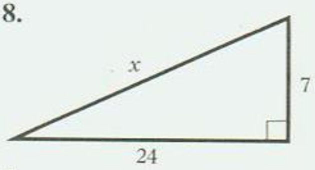
$r = \sqrt{s^2 + t^2}$

For exercises 7–12 use the information that is marked on the figure to find the value of x.

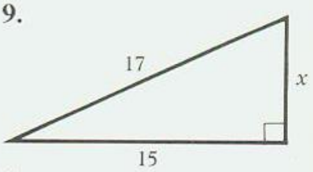
7.



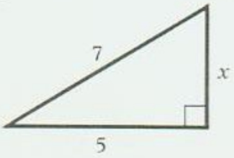
8.



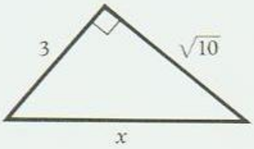
9.



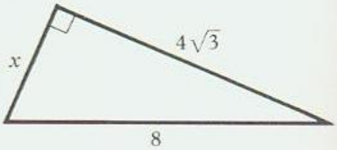
10.



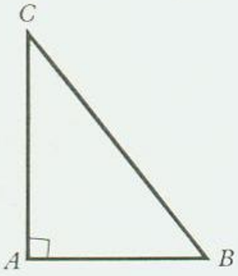
11.



12.



For exercises 13–18, use right triangle ABC.



13. If $AB = 6$, and $AC = 8$ then $BC =$?

14. If $BC = 15$, and $AB = 9$ then $AC =$?

15. If $AC = 2$ and $AB = 2$ then $BC =$?

16. If $BC = \sqrt{15}$ and $AB = \sqrt{10}$ then $AC =$?

17. If $AC = \sqrt{2}$ and $AB = \sqrt{3}$ then $BC =$?

18. If $AB = 2\sqrt{3}$ and $BC = 6$ then $AC =$?

B.

For exercises 19–24, decide whether or not the given numbers can be the lengths of sides for a right triangle.

19. 10, 24, 26

20. 20, 21, 29

21. 8, 15, 17

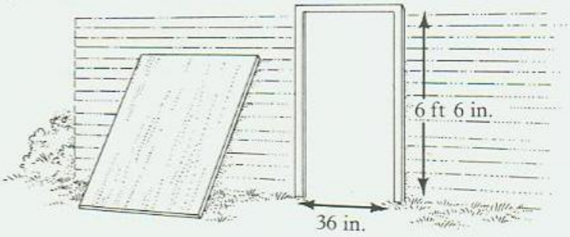
22. 7, 25, $\sqrt{674}$

23. 5, 13, $\sqrt{195}$

24. 5, 12, 13

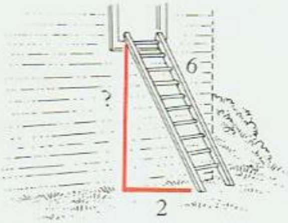
25. Find the length of the diagonal of a rectangle whose sides have lengths 10 and 18.

26. A door is 6 feet 6 inches tall and 36 inches wide. What is the widest sheet of plywood that can be carried through the door?


6 ft 6 in.
36 in.

(Ex. 26)

27. A 6-foot ladder is placed against a wall with its base 2 feet from the wall. How high above the ground is the top of the ladder?

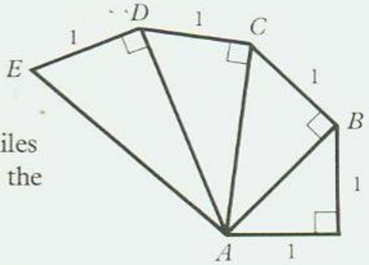


C.

28. Find the lengths of $\overline{AB}$, $\overline{AC}$, $\overline{AD}$, and $\overline{AE}$.

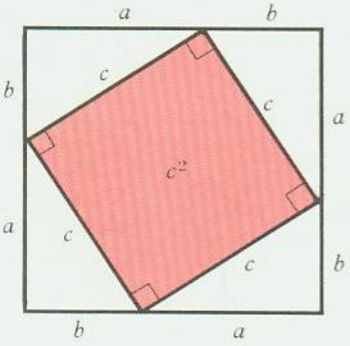
29. Construct with straightedge and compass segments with lengths $\sqrt{6}$ and $\sqrt{7}$.

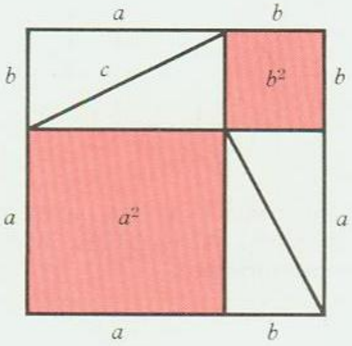
30. A person travels 8 miles due north, 3 miles due west, 7 miles due north, 11 miles due east. How far is that person from the starting point?



(Ex. 28)

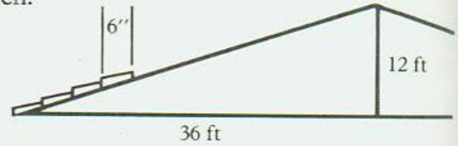
31. Use the squares with side length $a + b$ to show the Pythagorean Theorem is true.





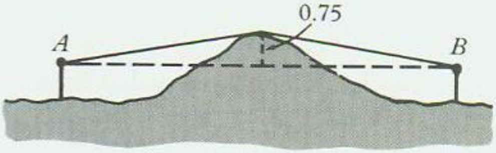
32. A 36-foot extension ladder must keep a 4-foot overlap when it is fully extended. If the base of the ladder is 10 feet from the building, how high up the building will the ladder reach?

33. (Use a calculator.)
If a row of shingles is 6 inches wide, how many rows of shingles are needed on each side of this roof.



(Ex. 33)

34. (Use a calculator.)
Surveying engineers want to measure the distance between two points A and B on rough land. They want to find the actual horizontal distance AB. If the earth is 0.75 meters higher midway between the two stakes, and if the measuring tape reads 27.0 meters, what is the actual distance AB?

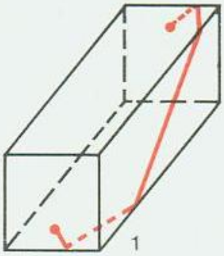


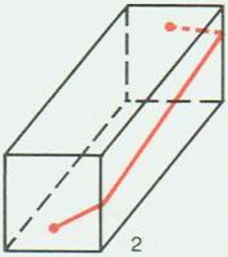
(Ex. 34)

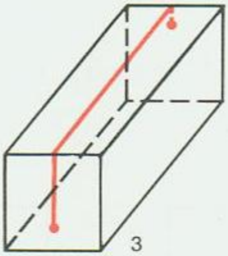
Activity

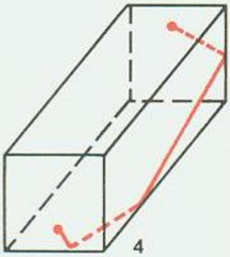
In a room 8 feet wide, 8 feet high, and 15 feet long, a fly crawls from the middle of the front wall, 1 foot above the floor, to the middle of the back wall, 1 foot below the ceiling.

1. Guess which one of the four paths below is the shortest one.


1

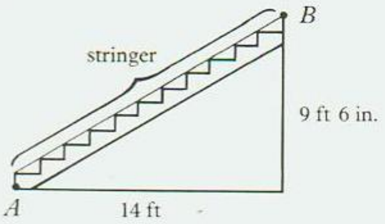

2


3

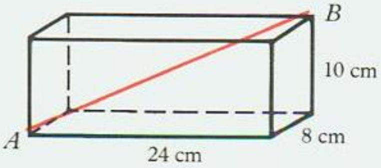

4

2. Make a box representing this room, draw the paths, and measure their lengths. (Any one of the diagrams in the PROBLEM SOLVING on the next page can be used to construct the box.)

35. A stairway will be built out of a piece of lumber called a *stringer*. (Refer to page 173, exercise 9.) The stringer runs from A to B. It covers a horizontal distance of 14 feet and a height of 9 feet 6 inches. What length piece of lumber must be bought for the stringer? (*Hint:* Lumber is sold in even foot lengths.)



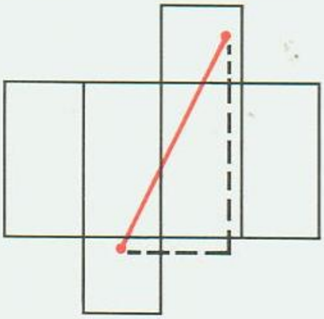
36. A box is 24 centimeters long, 8 centimeters wide, and 10 centimeters high. How long is the diagonal $\overline{AB}$?



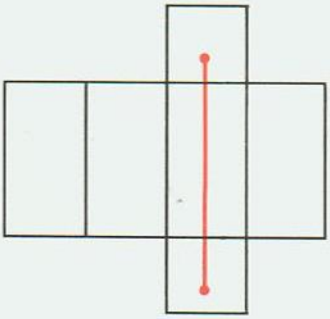
PROBLEM SOLVING

1. Imagine that the box from the Activity on page 230 has been cut apart. Match each of the four paths of the fly shown on page 230 with one of the "flattened out" diagrams below.

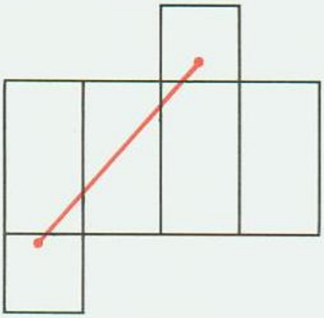
a



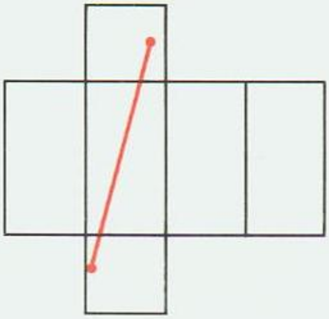
b



c



d

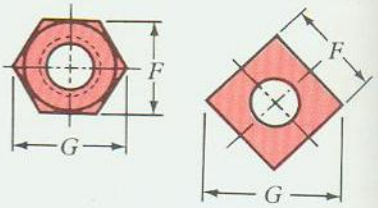


2. Complete a right triangle and calculate the length of each path.

7–2 Special Triangles

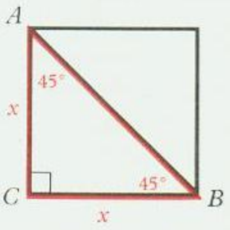
The figures and table on the right show the dimensions and specifications for two different kinds of machine nuts.

The dimension labeled F tells the size wrench needed for the nut. How can dimensions G be calculated? Knowledge of 45°-45°-90° and 30°-60°-90° triangles will help in this calculation.

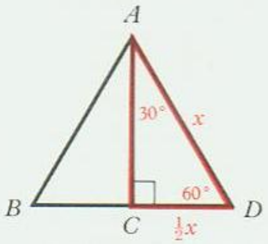


Square and Hexagon Machine Nuts

Nominal Size	F	G	
	Width Across Flats	Width Across Corners	
		Square	Hex.
	Basic	Max.	Max.
0	5/32	0.221	0.180
1	5/32	0.221	0.180
2	3/16	0.265	0.217
3	3/16	0.265	0.217
4	1/4	0.354	0.289



A 45°-45°-90° triangle is formed by 2 sides of a square and a diagonal.



A 30°-60°-90° triangle is formed by an altitude in an equilateral triangle.

Theorem 7–3 The length of the hypotenuse of a 45°-45°-90° triangle is $\sqrt{2}$ times the length of a leg.

Theorem 7–4 The length of the longer leg of a 30°-60°-90° triangle is $\frac{\sqrt{3}}{2}$ times the length of the hypotenuse or $\sqrt{3}$ times the length of the shorter side.

The proof of Theorem 7-3 is left as an exercise. We suggest a proof of Theorem 7-4 by applying the Pythagorean Theorem to $\triangle ACD$ above:

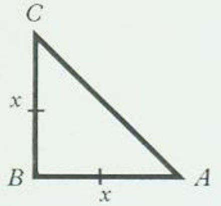
$$x^2 = (AC)^2 + \left(\frac{1}{2}x\right)^2$$

$$(AC)^2 = x^2 - \frac{x^2}{4} = x^2\left(1 - \frac{1}{4}\right)$$

$$AC = \sqrt{\frac{3}{4}x} = (\sqrt{3})\left(\frac{x}{2}\right) = \frac{\sqrt{3}}{2}(x)$$

These two theorems can be used to find unknown lengths in special triangles.

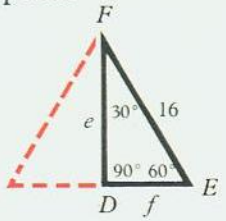
Example 1



1. $\angle A$ and $\angle C$ must be 45° angles.
2. By Theorem 7–3, $x\sqrt{2} = 12$

$$\text{so } x = \frac{12}{\sqrt{2}} \cdot \frac{\sqrt{2}}{\sqrt{2}} = \frac{12\sqrt{2}}{2} = 6\sqrt{2}$$

Example 2



1. $\overline{ED}$ is an altitude of an equilateral triangle so

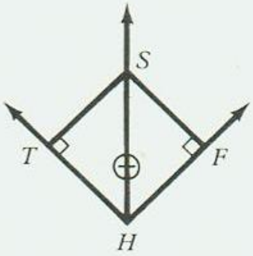
$$f = \frac{16}{2} = 8.$$

2. By Theorem 7–4, $e = \sqrt{3} \cdot f$
   or $e = 8\sqrt{3}$.

APPLICATION 1

How far is it from home plate to second base on a baseball diamond? Recall that the distance from home plate to first base is 90 feet. Since $\triangle HFS$ is a 45°-45°-90° triangle, Theorem 7-3 says that

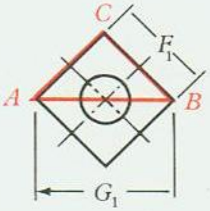
$$HS = \sqrt{2}(90) \doteq 127.28 \text{ feet.}$$

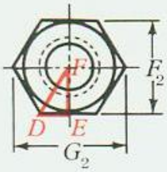


NOTE: $\sqrt{2}$ and $\sqrt{3}$ can only be approximated with a decimal number.

APPLICATION 2

The machine nuts used in the specification at the beginning of this lesson are square and regular hexagon in shape. Therefore $\triangle ABC$ is a 45°-45°-90° triangle and $\triangle DEF$ is a 30°-60°-90° triangle.





From Theorem 7-3, $G_1 = \sqrt{2}F_1$.

From Theorem 7-4, $FE = \sqrt{3}\,DE$

$$F_2 = 2FE = 2\sqrt{3}\,DE$$

$$DE = \frac{F_2}{2\sqrt{3}}$$

$$G_2 = 4\,DE = \frac{2}{\sqrt{3}}F_2.$$

For size 4 hexagonal machine nuts with $F_2 = \frac{1}{4}$, $G_2 = 4\,DE = \frac{2}{\sqrt{3}} \cdot \frac{1}{4} \doteq 0.289$.

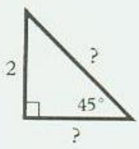
Notice that this is the value given in the table.

EXERCISES

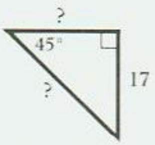
A.

For exercises 1–4 use Theorem 7–3 to find the lengths of the indicated sides.

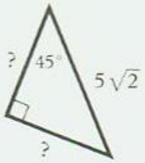
1.



2.



3.



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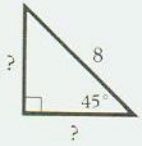

For exercises 5–12 use Theorem 7–4 to find the lengths of the sides of the triangle.

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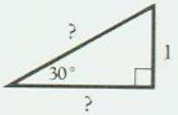

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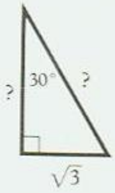

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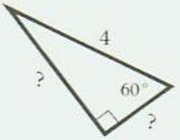

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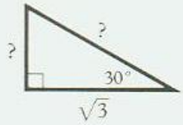

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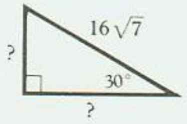

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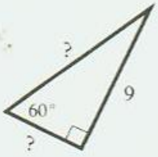

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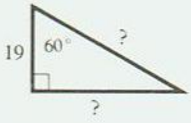

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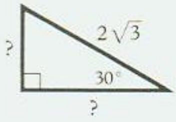
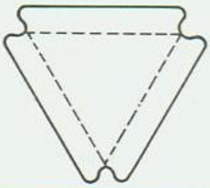
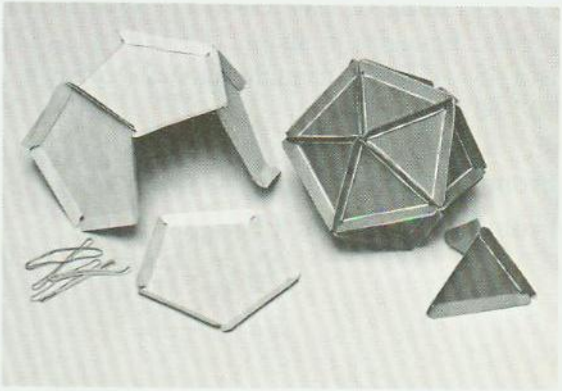


Activity

Use commercially-produced materials or cut from posterboard as many equilateral triangle patterns as you need and put them together with rubber bands as shown in the photograph.

There are eight convex solid figures that can be made using only equilateral triangle pieces of one size. How many of these can you make?



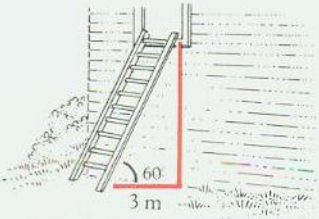


B.

13. In a right triangle one acute angle has a measure that is twice the measure of the other acute angle. If the length of the longer leg is 5, what is the length of the hypotenuse?

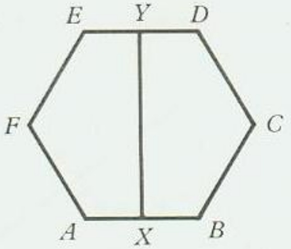
14. A window has a clear opening 41 inches wide and 26 inches high. Will a ping-pong table top 48 inches wide fit through the window?

15. A ladder leaning against a wall makes a 60° angle with the ground. If the base of the ladder is 3 meters from the building, how high above the ground is the top of the ladder?



16. Prove Theorem 7–3.

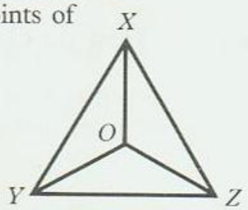
C.



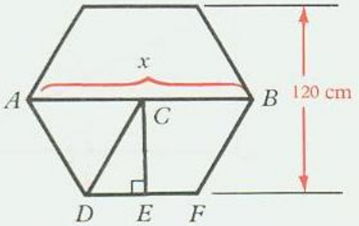
17. Prove that the altitude of an equilateral triangle with side of length s is $s\frac{\sqrt{3}}{2}$.

18. If the length of the side of a regular hexagon $ABCDEF$ is s, what is the length of $\overline{XY}$ if X and Y are midpoints of opposite sides?

19. If an equilateral triangle has side length s, find the radius of the circle that contains the three vertices.



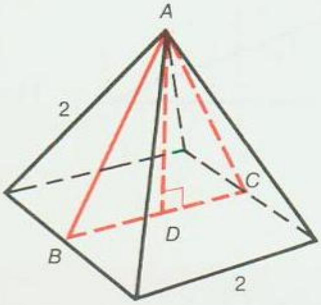
20. An architect is calculating the dimensions for a regular hexagon shaped window. If the height of the opening is 120 cm, find the width AB. ($m\angle ADF = 120$)



PROBLEM SOLVING

Consider a square-based pyramid whose edges all have length 2. Suppose points B and C are the midpoints of the indicated edges.

1. Find the lengths of $\overline{AB}$ and $\overline{AC}$.
2. Find the length of the altitude $\overline{AD}$ of the pyramid.
3. Is $\triangle ABC$ an equilateral triangle?

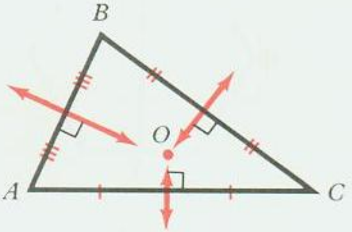


7–3 Triangle Concurrency Theorems

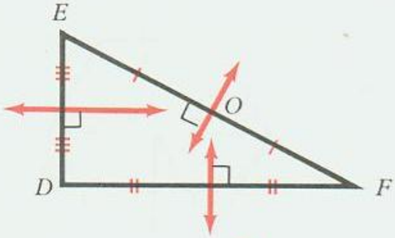
A manufacturer produces a product that is sold primarily in three major cities. A new factory is to be located an equal distance from each of the three cities. How can the location of the new factory be found?

The discussion below answers this question.

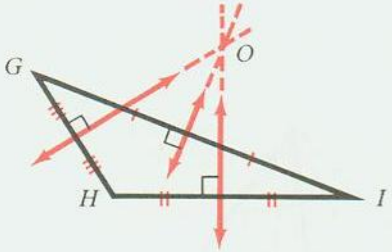
Construct the perpendicular bisectors of the sides of a triangle.



How do OA, OB, and OC compare?



How do OD, OE, and OF compare?

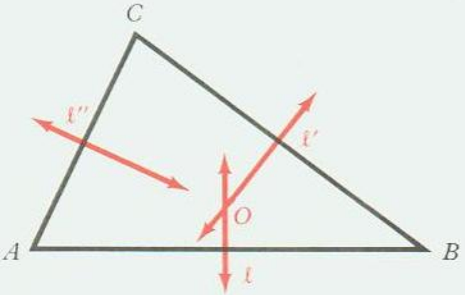


How do OG, OH, and OI compare?

Theorem 7–5 The perpendicular bisectors of the sides of a triangle intersect in a point O that is equidistant from the three vertices of the triangle.

PROOF

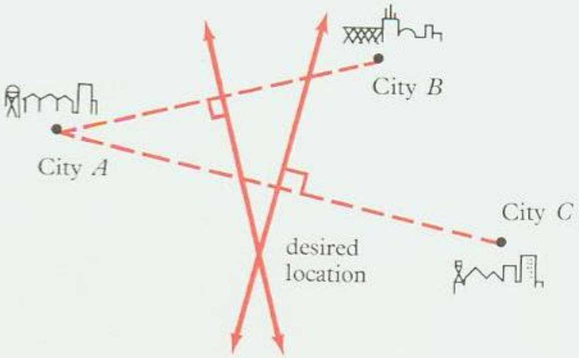
Given: $\triangle ABC$ with perpendicular bisectors ℓ, ℓ', and ℓ''.
Prove: ℓ, ℓ', and ℓ'' are concurrent in a point O and that $OA = OB = OC$.



Statements	Reasons
1. ℓ is the perpendicular bisector of $\overline{AB}$	**1.** Given
2. ℓ' is the perpendicular bisector of $\overline{BC}$	**2.** Given
3. ℓ and ℓ' intersect in a point O	**3.** If $\overline{AB} \nparallel \overline{BC}$ then $\ell \nparallel \ell'$.
4. $OA = OB$	**4.** A point on a perpendicular bisector is equidistant from the endpoints.
5. $OB = OC$	**5.** Why?
6. $OA = OC$	**6.** Transitive property of equality
7. O is on the perpendicular bisector of $\overline{AC}$.	**7.** A point equidistant from two points is on the perpendicular bisector of the segment determined by those points.
8. O lies on ℓ, ℓ', ℓ'' and $OA = OB = OC$.	**8.** Statements 4–8

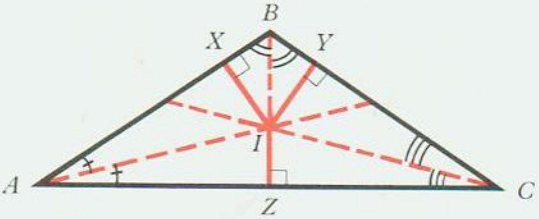
APPLICATION

At the beginning of the lesson we asked how a point can be located equidistant from three cities. Theorem 7-5 answers the question.



The perpendicular bisectors determine a point that is equidistant from the *vertices* of the triangle. We can also locate a point that is equidistant from the *sides* of the triangle.

In $\triangle ABC$ the three angle bisectors have been constructed.

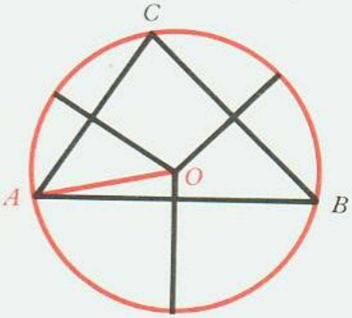


How do IX, IY, and IZ compare?

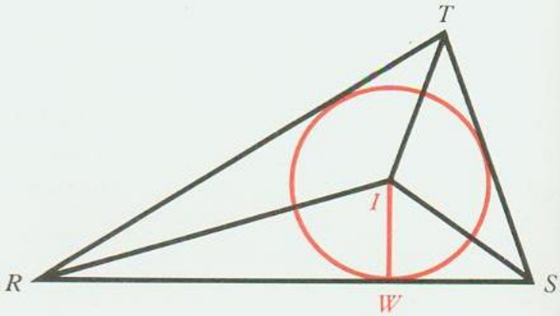
Theorem 7–6 The angle bisectors of the angles of a triangle are concurrent in a point I that is equidistant from the three sides of the triangle.

The points determined by the perpendicular bisectors (Theorem 7–5) and the angle bisectors (Theorem 7–6) are the centers of circles that have a special relationship with the triangle.

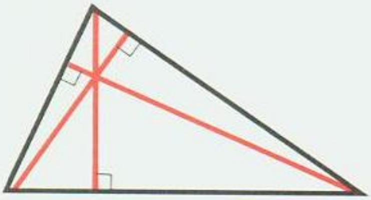
Circle O contains the three vertices of $\triangle ABC$. The center is the point of intersection of the perpendicular bisectors. The radius is OA. Circle O is called the *circumscribed circle.*



Circle I touches each side of $\triangle RST$ in exactly one point. The center is the point of intersection of the angle bisectors of the triangle. The radius is IW. Circle I is called the *inscribed circle.*

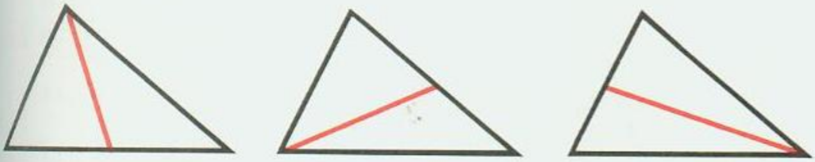


If you were to construct a triangle and its three altitudes, you would see that the lines containing the altitudes are concurrent.



Theorem 7–7 The lines that contain the altitudes of a triangle intersect in a point.

For any triangle there are three segments called *medians.*

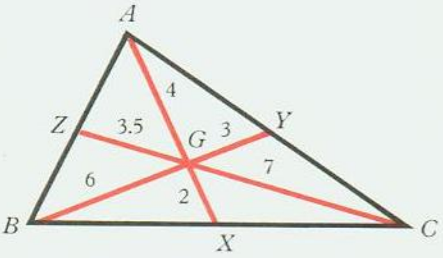


Definition 7-1

A **median** of a triangle is a segment joining a vertex to the midpoint of the opposite side.

If you were to construct the three medians of a triangle, you would see that the three medians are also concurrent.

There is another interesting relationship.



How do AG and AX compare?
How do BG and BY compare?
How do CG and CZ compare?

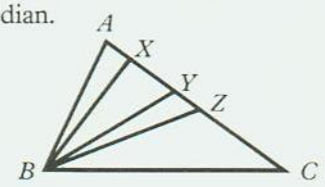
The following theorem is stated without proof.

Theorem 7–8 The medians of a triangle intersect in a point that is two thirds of the way from each vertex to the opposite side.

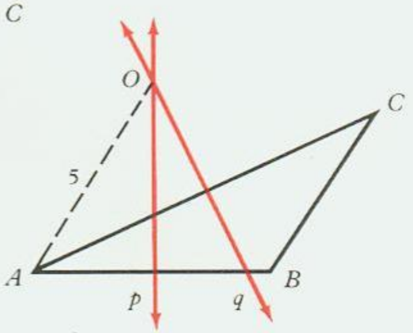
EXERCISES

A.

1. For $\triangle ABC$ name an altitude, an angle bisector, and a median.

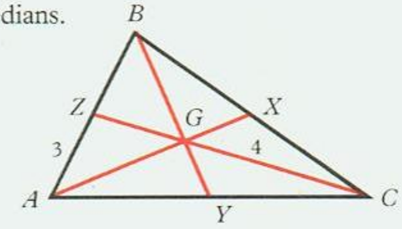


2. For $\triangle ABC$ the lines p and q are perpendicular bisectors of sides. If $OA = 5$, find OB and OC.

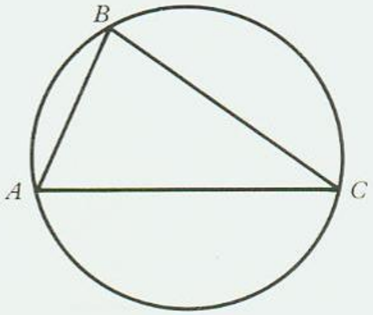


In $\triangle ABC$ $\overline{AX}$, $\overline{BY}$, and $\overline{CZ}$ are medians.

3. If $AZ = 3$, what is ZB?
4. If $CG = 4$, what is GZ?
5. If $AB = BY$, what is BG?



6. Trace $\triangle ABC$ and the circle that passes through the points A, B, and C. Use a compass and straight edge to find the center of the circle. (*Hint:* Use Theorem 7-5.)



Exercises 7–12 ask for a construction to be completed. Use a straight edge and compass or Mira for these constructions. Each exercise should fill nearly a full sheet of paper.

7. Illustrate Theorem 7-5 by constructing a triangle and all three perpendicular bisectors.

8. Illustrate Theorem 7–6 by constructing a triangle and all three angle bisectors.

9. Illustrate Theorem 7–7 by constructing a triangle and all three altitudes.

10. Illustrate Theorem 7–8 by constructing a triangle and all three medians.

11. Draw a triangle and construct the circumscribed circle. (*Hint:* Theorem 7–5 tells how to find the center of this circle.)

12. Construct $\triangle ABC$ with sides 14 cm, 17 cm, and 20 cm. Very accurately construct the intersections of the medians, altitudes, and perpendicular bisectors of the sides. (Your construction should verify that these three points lie on a line.)

Discover the answers to exercises 13–16 by experimenting with sketches. Fill in the blank with the word "always," "sometimes," or "never."

13. The altitudes of a triangle _?_ intersect inside the triangle.

14. The medians of a triangle _?_ intersect outside the triangle.

15. The perpendicular bisectors of the sides of a triangle _?_ intersect inside the triangle.

16. The angle bisectors of a triangle _?_ intersect inside the triangle.

17. The center of the circumscribed circle _?_ lies inside the triangle.

18. The center of the inscribed circle _?_ lies inside the triangle.

In exercises 19 and 20 fill in the first blank with one of the words "acute," "right," or "obtuse," and fill in the second blank with one of the words "inside," "on," or "outside."

19. The point of intersection of the lines containing the altitudes of a _?_ triangle is _?_ the triangle.

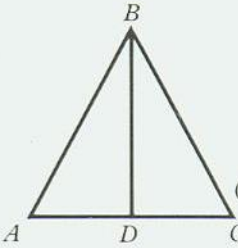
20. The point of intersection of the perpendicular bisectors of the sides of a _?_ triangle is _?_ the triangle.

B.

21. Draw a triangle and construct the inscribed circle.

22. Prove that the median from the vertex angle of an isosceles triangle is also an altitude.

23. Prove that the median from the vertex angle of an isosceles triangle is also the angle bisector.



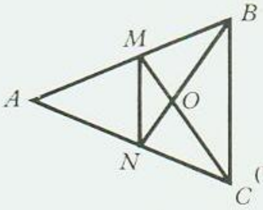
(Exs. 22, 23)

24. Prove that an altitude of an equilateral triangle is also a median of the triangle.

25. Prove that a median of an equilateral triangle is also an altitude of the triangle.

26. Prove that the medians from the base angles of an isosceles triangle are congruent.

27. In $\triangle ABC$, $AB = AC$, and $\overline{BN}$ and $\overline{CM}$ are angle bisectors. Prove that $\triangle MON$ is an isosceles triangle.



(Ex. 27)

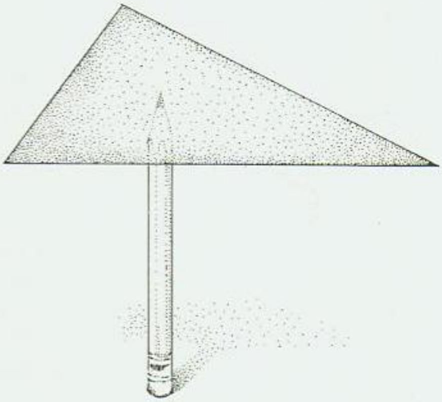
Activity

Centroid means center of mass. If a triangle could be "balanced," its "balance point" would be its centroid.

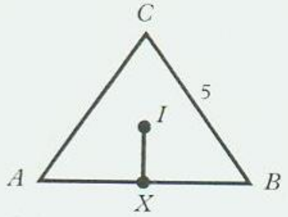
Experiment: Cut out a large triangular shape from a piece of cardboard or a thin piece of wood. (Your triangle should be scalene.)

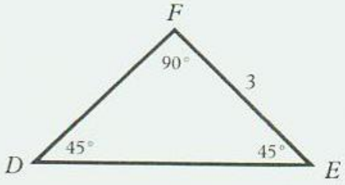
Find the centroid of the figure.

Can you balance the object from that point? Can you spin the object?



C.





28. If $\triangle ABC$ is equilateral, find the length IX of the radius of the inscribed circle.

29. If $\triangle DEF$ is a 45°-45°-90° triangle as shown, find the length of the radius of the circumscribed circle.

30. Prove that the point of intersection of the angle bisectors of an isosceles triangle is on the altitude from the vertex angle.

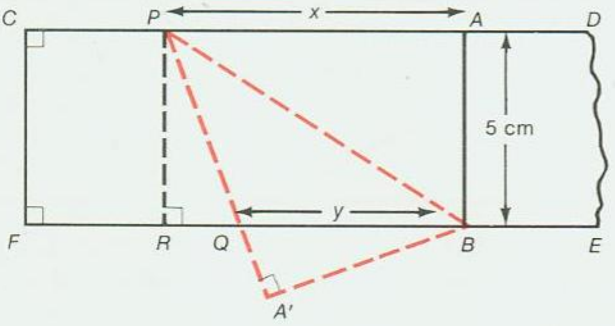
31. Prove that the three medians of an equilateral triangle divide the triangle into six congruent triangles.

32. Prove that in an equilateral triangle the perpendicular bisectors of the sides, the angle bisectors, the altitudes, and the medians all intersect in the same point.

33. Prove that the altitudes from the base angles of an isosceles triangle are congruent.

PROBLEM SOLVING

$CDEF$ is a strip of paper 5 cm wide and $\overline{AB}$ is a crease parallel to $\overline{CF}$.



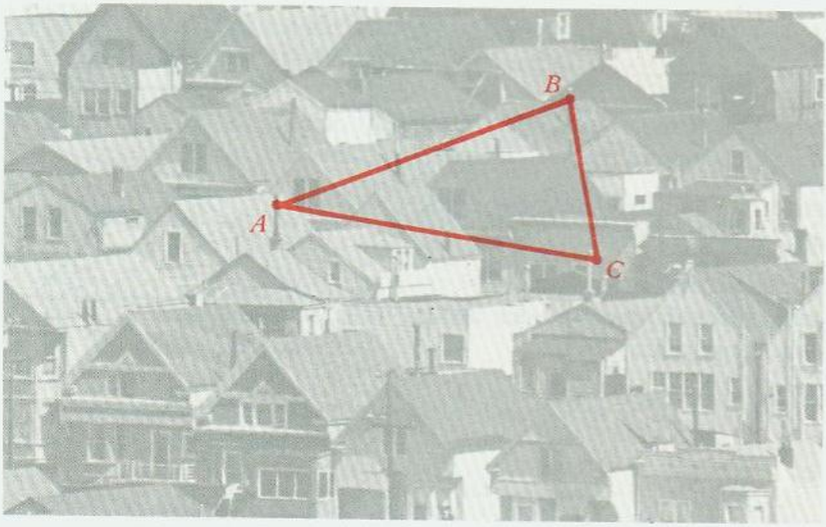
Let P be any point on $\overline{CA}$ and fold along $\overline{PB}$ locating point Q.

Let $PA = x$ and $QB = y$.

1. Show that $PQ = QB = y$.
2. Show that x and y are related by $y = \dfrac{x^2 + 25}{2x}$.

   (*Hint:* Show that $\triangle PQR \cong \triangle BQA'$ and use the Pythagorean Theorem.)

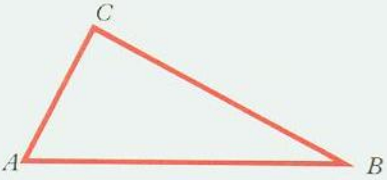
7–4 Triangle Inequality



We often hear phrases like "The shortest distance between two points is a straight path," or "It is 100 m the way a crow flies." These expressions are informal ways of describing an important relationship that we shall state as a postulate.

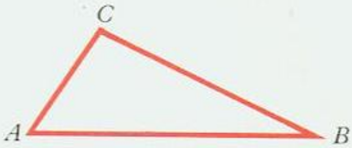
The following three examples suggest the postulate.

Measure the sides.



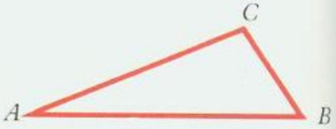
Observe that
$AB < AC + CB$.

Measure the sides.

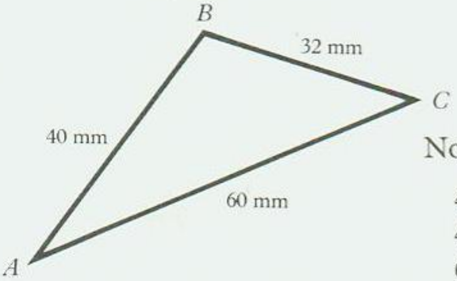


Observe that
$AC < AB + CB$.

Measure the sides.



Observe that
$CB < AB + AC$



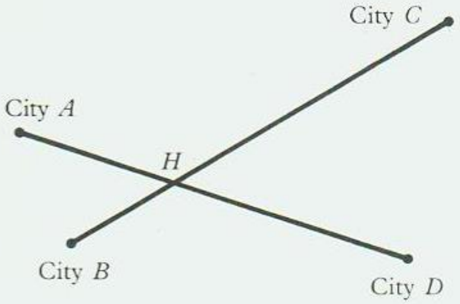
Notice that

$$40 + 32 > 60$$
$$40 + 60 > 32$$
$$60 + 32 > 40$$

Triangle Inequality Postulate

The sum of the lengths of two sides of a triangle is greater than the length of the third side.

APPLICATION 1



A railroad company is to build an engine house to service four cities located at the vertices $ABCD$ of a quadrilateral as shown here. Where should the engine house H be located so that the length, and hence the construction costs, of the roadbed $AH + BH + CH + DH$ is a minimum?

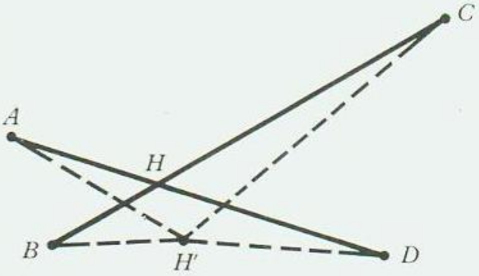
Answer. At the point of intersection of the diagonals of $ABDC$.

Suppose H' is some other location. Then by the Triangle Inequality Postulate

1. $BH + CH < BH' + CH'$ in $\triangle BCH'$,
2. $AH + DH < AH' + DH'$ in $\triangle ADH'$.

Therefore,

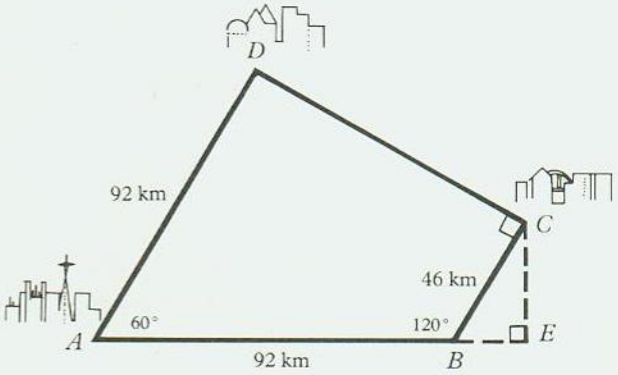
$AH + BH + CH + DH < AH' + BH' + CH' + DH'$.



APPLICATION 2

If cities are located at points A, B, C, and D as shown in the figure, what is the minimum length track from a centrally located point H to each of the cities?

Answer. From Application 1 we see that the minimum length track is the sum of the lengths of the diagonals $\overline{AC}$ and $\overline{BD}$. To find the length of these diagonals, first we must find BE and CE. Since the length of the shorter leg of a 30°-60°-90° triangle is half the length of the hypotenuse, then $BE = 23$ km. The longer leg, CE, has a length of $23\sqrt{3}$ km or approximately 39.84 km.



Applying the Pythagorean Theorem to $\triangle AEC$,

$$AC = \sqrt{AE^2 + CE^2}$$

$$= \sqrt{13{,}225 + 1587} \doteq 121.7 \text{ km}.$$

$$BD = 92 \text{ km}.$$

Therefore, the minimum length is approximately $121.7 \text{ km} + 92 \text{ km} = 213.7$ km.

EXERCISES

A.

In exercises 1–7 decide whether or not the given set of numbers could be the lengths of the sides of a triangle.

1. $\{4, 5, 7\}$

2. $\{4, 5, 17\}$

3. $\{6, 13, 7\}$

4. $\{9, 13, 17\}$

5. $\{7, 7, 13\}$

6. $\{j, k, j + k\}$

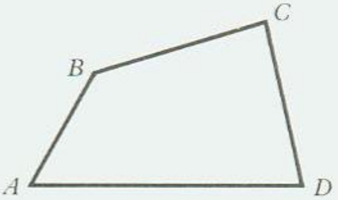
7. $\{a, 3a, 3a\}$

B.

8. If two sides of a triangle have lengths 2 and 5, then the length of the third side is greater than $\underline{?}$, and less than $\underline{?}$.

9. If the lengths of two sides of a triangle are 7 and 9, what are the possible lengths of the third side?

10. Prove that for any quadrilateral $ABCD$, $AB + BC + CD > AD$.



(Ex. 10)

Activity

Construct a triangle with sides of length 8 cm, 11 cm, and 15 cm, and construct the following points.

a. The midpoint of the three sides

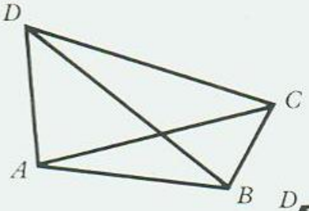
b. The feet of the three altitudes

c. The intersection of the three altitudes. Label this point *H*.

d. The midpoints of the segments joining *H* with the vertices of the triangle

How do the nine points constructed in parts a, b, and d appear to be related?

11. Prove that the shortest path between two points A and B is the segment joining them.

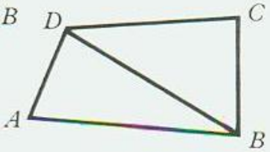


C.

12. Prove that the sum of the lengths of the sides of a quadrilateral is greater than the sum of the lengths of the diagonals.

13. Prove that the length of a diagonal of a quadrilateral is less than one half the perimeter. That is:

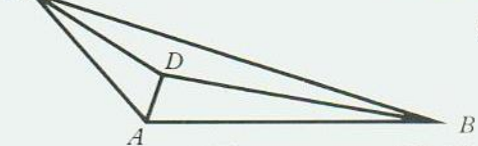
$$BD < \frac{AB + BC + CD + AD}{2}.$$



(Ex. 13)

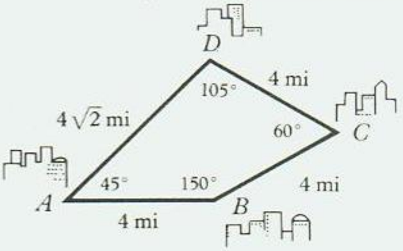
14. Prove that if D is a point in the interior of $\triangle ABC$, then

$$\tfrac{1}{2}(AB + BC + AC) < AD + BD + CD.$$



(Ex. 14)

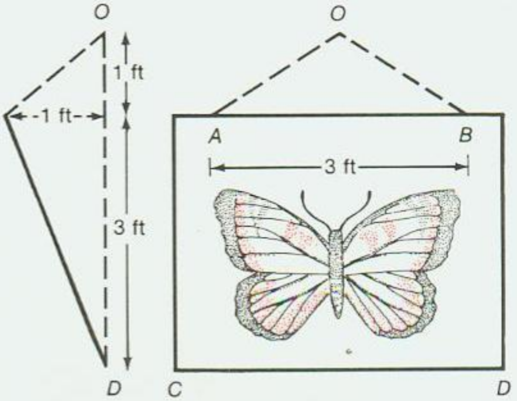
15. Suppose cities are located at points A, B, C, and D as shown in this figure. Where is a point H located if $AH + BH + CH + DH$ is a minimum? What is the value of this minimum? (Use a calculator.)



PROBLEM SOLVING

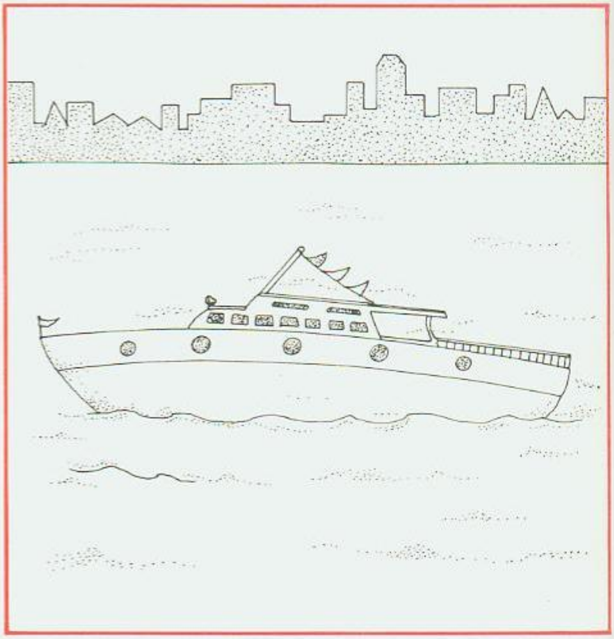
This sketch shows a front and side view of a picture that is supported against the wall along its bottom $\overline{CD}$, and slants away from the wall at the top. It is supported at the top with a wire AOB, O being a nail on the wall.

Find the length of the wire AOB.

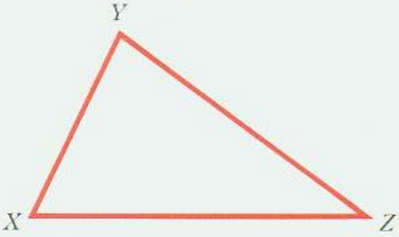


7–5 Inequalities in a Triangle

Imagine a physical setting involving rather large distances that cannot be measured directly. Sometimes the actual distance does not need to be known but only a comparison between two distances. For example, a ship may want to compare its distance from a point on shore to its distance from an island, in order to make sure it is staying outside an imaginary line halfway between the shore and the island. The theorem in this lesson will provide one method for doing this.

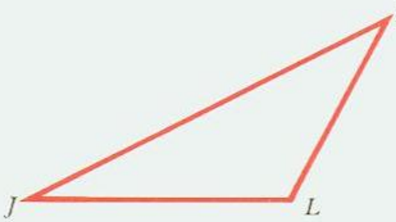


$m\angle Z < m\angle X$



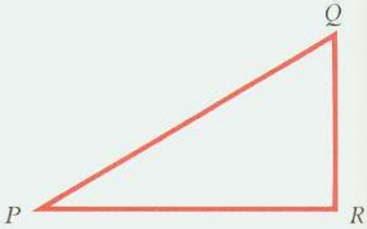
Which is shorter, $\overline{XY}$ or $\overline{YZ}$?

$m\angle J < m\angle K$



Which is shorter, $\overline{JL}$ or $\overline{LK}$?

$m\angle P < m\angle Q$

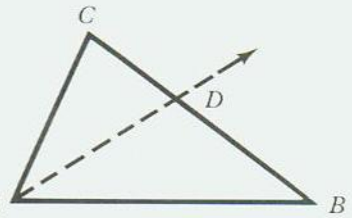


Which is shorter, $\overline{PR}$ or $\overline{QR}$?

Theorem 7–9 If the measures of two angles of a triangle are unequal, then the length of the side opposite the smaller angle is less than the length of the side opposite the larger angle.

PROOF

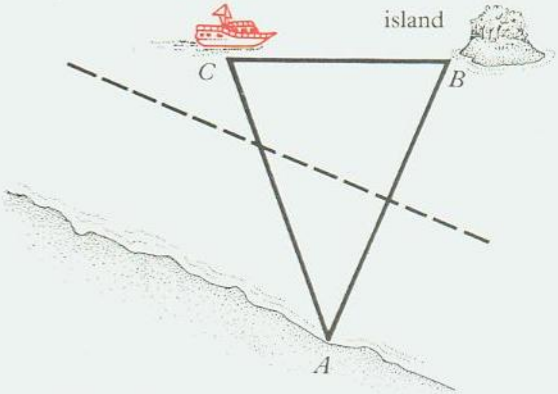
Given: $\triangle ABC$ with $m\angle B < m\angle A$
Prove: $AC < BC$



Statements	Reasons
1. $m\angle B < m\angle A$	1. Given
2. There exists a point D on $\overrightarrow{BC}$ so that $m\angle BAD = m\angle B$.	2. Protractor Postulate
3. $\overline{AD} \cong \overline{BD}$	3. If two angles of a triangle are congruent, then the sides opposite them are congruent.
4. $AD = BD$	4. Why?
5. $AC < AD + DC$	5. Why?
6. $AD + DC = BD + DC$	6. Addition of equals property
7. $BD + DC = BC$	7. Definition of between for points
8. $AC < BC$	8. Substitution Principle

APPLICATION

How can a ship determine that its position is outside an imaginary boundary line that is halfway between the shore and an island?



Answer. Persons located at both points A and B determine, by using radio, radar, or laser beams, that $m\angle A < m\angle B$. When this information is radioed to the ship, the ship's captain concludes from Theorem 7-9 that $CB < CA$.

The converse of Theorem 7-9 is stated here without proof.

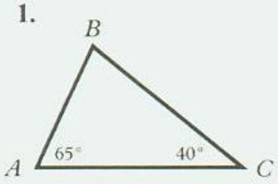
Theorem 7–10 If the lengths of two sides of a triangle are unequal, then the measure of the angle opposite the shorter side is less than the measure of the angle opposite the longer side.

EXERCISES

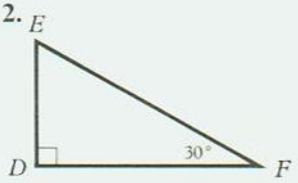
A.

In exercises 1–4 name the shortest side and the longest side of the given triangle.

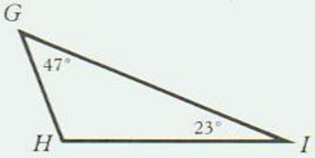
1.



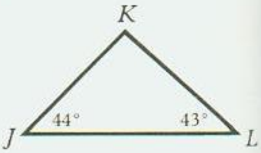
2.



3.



4.



In exercises 5–8 list the sides from shortest to longest for a triangle $\triangle ABC$ if:

5. $m\angle A = 46, m\angle B = 30$

6. $m\angle C = 101, m\angle B = 70.$

7. $m\angle A = 59, m\angle C = 61.$

8. $m\angle B = 48, m\angle A = 47.$

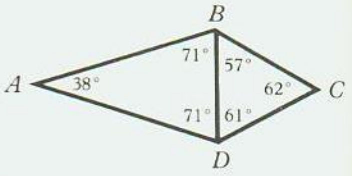
In exercises 9–10 list the angles from the one with smallest measure to one with largest measure for $\triangle ABC$ if:

9. $AB = 17, BC = 21, AC = 18.$

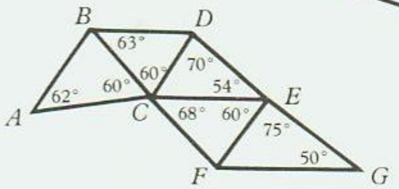
10. $AB = 15, AC = 16, BC = 17$

B.

11. List the sides of the quadrilateral from shortest side to longest side.

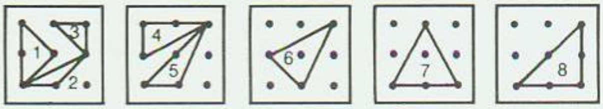


12. List all the segments in this figure from the one with shortest length to the one with the longest length. (Assume that all the indicated angle measures are correct.)



Activity

Use a 3-by-3 square array of dots. There are eight triangles different in size and shape that can be drawn.

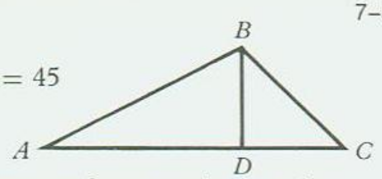


There are over 30 different triangles that can be drawn on a 4 by 4 square array of dots. Draw as many of these triangles as you can.

Suggestion: Be systematic. Do not start by drawing at random. (If dot paper is not available, graph paper can be used.)

13. **Given:** $m\angle DBC = m\angle BCD = 45$

Prove: $CD < AB$



14. Prove that the perpendicular segment from a point to a line is the shortest segment from the point to the line.

C.

15. Prove that the median from vertex Y of scalene triangle XYZ is longer than the altitude from vertex Y.

16. Prove that the sum of the lengths of the three altitudes of $\triangle ABC$ is less than the sum of the lengths of the sides of the triangle.

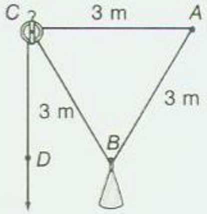
17. The three main work centers in the kitchen are the refrigerator, sink, and stove. You can picture them as the points of a triangle. One rule of thumb is: The three sides of the "kitchen triangle" should add up to more than 12 feet and less than 22 feet. Also, the shortest side of the triangle should be between the sink and the stove.

Below is a chart of possible "kitchen triangles." First decide whether each triangle is possible. Then state whether each triangle follows the rule of thumb.

	Stove-Sink	Stove-Refrigerator	Refrigerator-Sink
a.	5 feet	4 feet	8 feet
b.	10 feet	11 feet	11 feet
c.	6 feet	8 feet	7 feet
d.	3 feet	7 feet	4 feet
e.	3 feet	8 feet	4 feet

PROBLEM SOLVING

Suppose that *ABCD* is a flexible wire. *A* is a fixed point and *C* is a fixed pulley. *B* is a weight that slides freely along the wire so that *AB* and *CB* are always equally inclined to the vertical. Find how far *B* rises if *D* is pulled down 2 meters.



Important Ideas—Chapter 7

Terms

Median (p. 239)

Postulates

Triangle Inequality Postulate. The sum of the lengths of two sides of a triangle is greater than the length of the third side.

Theorems

7-1 **Pythagorean Theorem.** If $\triangle ABC$ is a right triangle, then the square of the length of the hypotenuse equals the sum of the squares of the lengths of the legs.

7-2 If $\triangle ABC$ has side lengths a, b, and c, and $c^2 = a^2 + b^2$, then $\triangle ABC$ is a right triangle.

7-3 The length of the hypotenuse of a 45°-45°-90° triangle is $\sqrt{2}$ times the length of a leg.

7-4 The length of the longer leg of a 30°-60°-90° triangle is $\frac{\sqrt{3}}{2}$ times the length of the hypotenuse or $\sqrt{3}$ times the length of the shorter side.

7-5 The perpendicular bisectors of the sides of a triangle intersect in a point O that is equidistant from the three vertices of the triangle.

7-6 The angle bisectors of the angles of a triangle are concurrent in a point I that is equidistant from the three sides of the triangle.

7-7 The lines that contain the altitudes of a triangle intersect in a point.

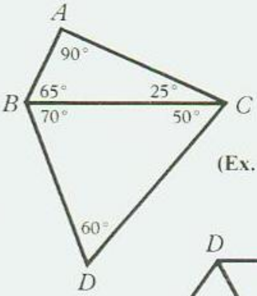
7-8 The medians of a triangle intersect in a point that is two thirds of the way from each vertex to the opposite side.

7-9 If the measures of two angles of a triangle are unequal, then the length of the side opposite the smaller angle is less than the length of the side opposite the larger angle.

7-10 If the lengths of two sides of a triangle are unequal, then the measure of the angle opposite the shorter side is less than the measure of the angle opposite the longer side.

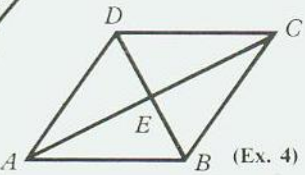
Chapter 7—Review

1. Indicate whether the following problems are true or false.
   - **a.** A triangle can have side lengths of 2 cm, 3 cm, and 5 cm.
   - **b.** In $\triangle ABC$, $m\angle A = 120$, $m\angle B = 20$, and $m\angle C = 40$. The longest side is $\overline{BC}$.
   - **c.** A rectangle with dimensions 10 cm by 24 cm has a 26 cm diagonal.
   - **d.** If the longer leg of a 30°-60°-90° triangle has a length of $3\sqrt{3}$, then the hypotenuse has length of 6.
2. What is the longest side of the figure given the indicated angle measures? (Figure is not drawn to scale.)



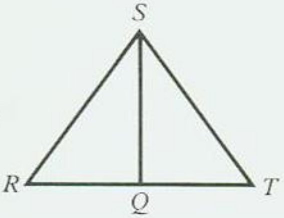
(Ex. 2)

3. Find the length of a leg of an isosceles right triangle if the length of the hypotenuse is 4 cm.
4. If $\overline{AB} \cong \overline{BC} \cong \overline{CD} \cong \overline{AD}$, and $\overline{AC} \perp \overline{BD}$ with $BD = AB = 2$, find AC.



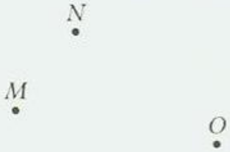
(Ex. 4)

5. Prove that the median $\overline{SQ}$ of isosceles triangle RST is also the perpendicular bisector of the base.



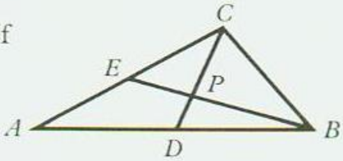
(Ex. 5)

6. Locate the point that is equal distance from points M, N, and O.



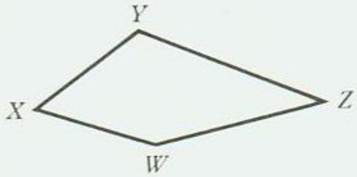
(Ex. 6)

7. If a 20 foot ladder is placed so that the foot of the ladder is 12 feet from a wall, how high above the ground will the ladder reach?
8. E and D are midpoints of $\overline{AC}$ and $\overline{AB}$. If $EP = 4$, find EB.



(Ex. 8)

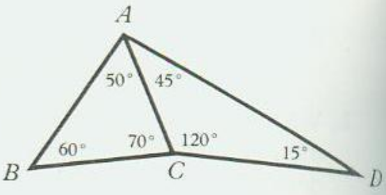
9. **Given:** $\overline{YZ}$ is the longest side of quadrilateral $XYZW$. $\overline{XW}$ is the shortest side.
   Prove: $m\angle X > m\angle Z$



Chapter 7—Test

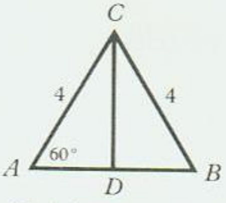
1. Indicate whether the following are true or false.
   - **a.** A triangle can have side lengths of 6, 6, and 13.
   - **b.** If a triangle has side lengths of 16, 30, and 44, then the triangle is a right triangle.
   - **c.** If the vertex angle of an isosceles triangle is 30°, then the base of the triangle is the shortest side.
   - **d.** If the longer leg of a $30°$-$60°$-$90°$ triangle has length 2, then the length of the hypotenuse is $\frac{2\sqrt{3}}{3}$.

2. What is the shortest side of the figure, given the indicated angle measures? (Figure is not drawn to scale.)



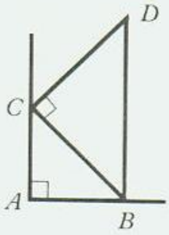
(Ex. 2)

3. In the given figure, $\overline{CD}$ is perpendicular to $\overline{AB}$. If $AC = 4$, find AD and CD.



(Ex. 3)

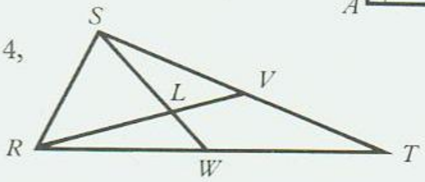
4. $\triangle ABC$ and $\triangle BCD$ are isosceles right triangles. If $AC = 2$, find BD.



(Ex. 4)

5. Prove that the angle bisector of the vertex angle of an isosceles triangle is also a median.

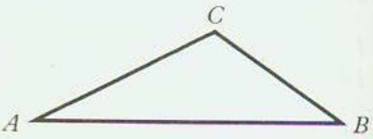
6. $\overline{SW}$ and $\overline{RV}$ are medians of $\triangle RST$. $SL = 4$, $SW = 6$, and $RV = 9$. Find RL.



(Ex. 6)

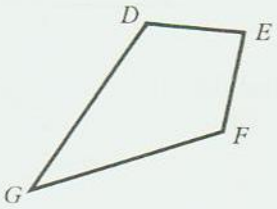
7. If a brace for a wall is placed 80 cm from the wall and it touches the wall 150 cm from the floor, how long is the brace?

8. In $\triangle ABC$ locate the point that is equidistant from sides $\overline{AB}$, $\overline{BC}$, and $\overline{AC}$.



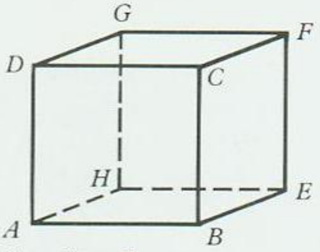
(Ex. 8)

9. **Given:** $DE = EF$, $DG = FG$
   $DE < DG$
   Prove: $m\angle G < m\angle E$

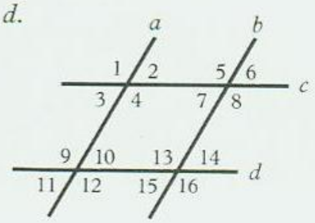


Cumulative Review—Chapters 4–7

1. Use the figure of a cube to identify the following:
   a. a line parallel to plane $ABEH$
   b. a line skew to $\overleftrightarrow{DC}$
   c. a plane perpendicular to $\overleftrightarrow{CB}$
   d. a line parallel to $\overleftrightarrow{GC}$
   e. a plane parallel to plane $CFEB$.

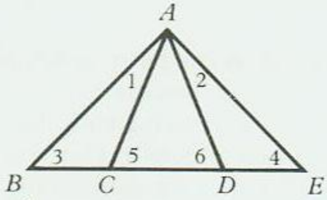


2. Assume line a is parallel to line b and line c is parallel to line d.
   a. If $m\angle 1 = 105$, find $m\angle 14$.
   b. If $m\angle 5 = 2x - 10$ and $m\angle 10 = x + 5$, find $m\angle 5$ and $m\angle 10$.
   c. If $m\angle 12$ is 10 more than twice the measure of $\angle 11$, find $m\angle 12$ and $m\angle 11$.



3. **Given:** $\angle 1 \cong \angle 2$, $\angle 3 \cong \angle 4$
   Prove: $\triangle ABC \cong \triangle AED$

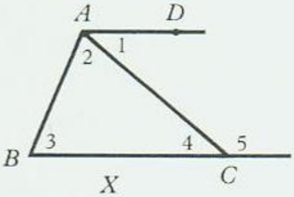
4. **Given:** $\overline{DE} \cong \overline{BC}$, $\overline{AD} \cong \overline{AC}$
   Prove: $\triangle BDA \cong \triangle ECA$



(Exs. 3, 4)

5. If $\angle ACB$ is the vertex angle of isosceles triangle ABC and $m\angle 2 = 70$, find $m\angle 5$.

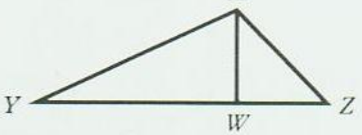
6. If $\overline{AD} \parallel \overline{BC}$, $m\angle 3 = 45$, $m\angle 1 = 65$, find $m\angle 2$.



(Exs. 5, 6)

7. If $\overline{XW}$ is perpendicular to $\overline{YZ}$, $m\angle Y = 30$, $m\angle Z = 45$, and $XW = 3$, find the lengths of $\overline{XY}$, $\overline{YW}$, and $\overline{XZ}$.

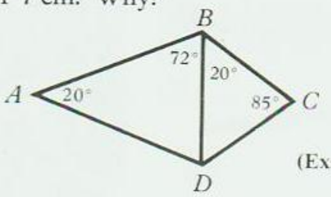
8. If $\overline{XZ}$ is an altitude to $\overline{XY}$, $XY = 12$, and $XZ = 5$, find YZ.



(Exs. 7, 8)

9. If two shorter sides of a triangle have lengths of 5 cm and 3 cm, is it possible for the longest side to have a length of 7 cm? Why?

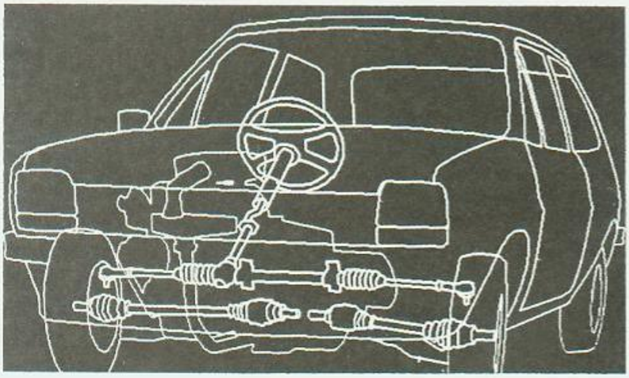
10. **Given:** Quadrilateral $ABCD$
    Find: a. $m\angle ADB$
    b. $m\angle BDC$
    c. $m\angle ADC$



(Exs. 10, 11)

11. List the sides of quadrilateral $ABCD$ from shortest side to longest side.

Computer Graphics: Computer Aided Design

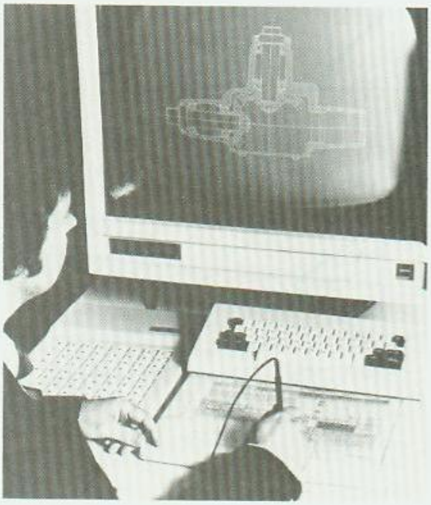


Computer graphics has revolutionized the job of an industrial designer. The designer must be able to imagine and analyze complex shapes composed of geometric figures. With computer-aided-design (CAD) the designer uses the computer to draw the designs.

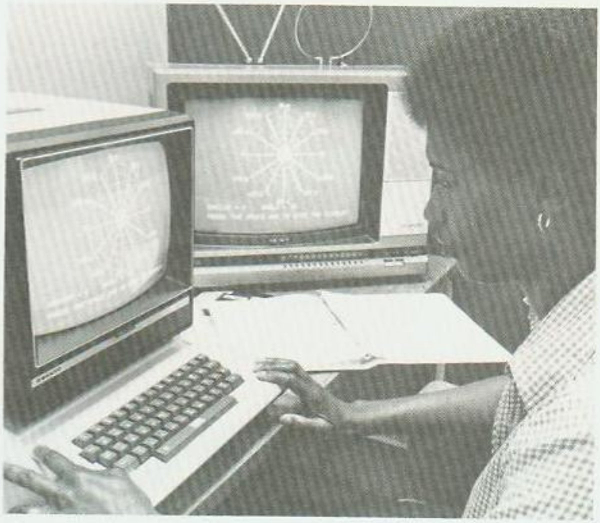
The photograph shows an automobile design generated by a computer. The computer can display this design in an unlimited number of positions. The view from the top, bottom, side, front, or back can be displayed on the screen.

The photographs below show designers at computer terminals. The designer on the right is using an electronic pen to alter the dimensions of a section template. By touching the screen with the pen, he is communicating with the computer.

The designer on the left is working with a cross-section of a 3-dimensional model. The table in front of him is used to give the computer commands. By placing the electronic pen on different squares, he can add or delete portions of the drawing. He can ask for an enlargement of part of the drawing. The computer library includes drawings for standard parts, such as bearings and gears. These parts can be added to the design.



Computer graphics has recently been more available to students through the use of microcomputers. On one popular microcomputer model the screen is divided into 280 (invisible) rows and 160 (invisible) columns. For example, the small space that lies in the fortieth row and twentieth column is the point named (40, 20). By lighting a large set of points, lines appear on the screen. For example, the following program displays a rectangle on the screen whose corners are the points (40, 20), (220, 20), (220, 120), and (40, 120).



The program, or list of instructions to the computer, is shown below. Line 10 tells the computer to prepare to receive instructions about graphics (HGR). In line 20, HPLOT is an instruction to draw a line on the screen from the point at (40, 20) to the point at (220, 20). Check the other instructions to see how the rectangle was drawn.

```
10  HGR
20  HPLOT 40, 20 TO 220, 20
30  HPLOT 40, 20 TO 40, 120
40  HPLOT 40, 120 TO 220, 120
50  HPLOT 220, 20 TO 220, 120
60  END
```

1. Write the HPLOT commands that will form a square on the computer screen.
2. Write four HPLOT commands that will form a rectangle twice as high as wide.
3. Create a figure that consists of line segments. Then write HPLOT commands that will put your figure on the TV screen.